The Electronic Journal of Mathematics and Technology, Volume 19, Number 2, ISSN 1933-2823

Geometric Loci Analysis Through Automated Reasoning Tools in GeoGebra: A case study

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Abstract

A common response to new educational technology is to suggest banning it, arguing that it could replace the development of certain skills and knowledge with the capabilities of the new tool. To counter this argument, it is essential to provide examples demonstrating how the wise use of new instruments can enhance the teaching and learning of mathematical competencies.

In our present contribution, we address the situation described above through an example in Geometry which incorporates the following elements:

a) the automated reasoning tools of GeoGebra Discovery, an experimental version of the mathematical software GeoGebra;

b) the development of mathematical reasoning and proof competencies through elementary geometry problems, such as loci computation; and

c) a concrete geometric construction as triggering event: given a triangle ABC, find the locus of points P such that $\angle ABP$ and $\angle ACP$ are congruent.

This construction can be quickly done using GeoGebra Discovery, but what does "finding" mean here? Is it just creating a visual image or finding an equation with coefficients based on the positions of A, B, and C? Our goal is to understand the geometric locus both symbolically and geometrically. As we explore with the help of algebra and geometry software, we'll discover various connections to geometric concepts that will deepen our understanding of elementary geometry.

In summary, our goal is to describe the challenges that arise in this elementary, yet highly inspiring and intriguing context, as an example of the methodological protocols and clear advantages associated with new technologies in mathematics education.

1 Introduction

This article is an expanded version of the communication "Automated reasoning tools for dealing with elementary but intriguing geometric loci" presented at the *Asian Technology Conference in Mathematics 2024* (see [15]).

1.1 On a geometric construction related to an Olympiad problem

In the field of Mathematics Education, the introduction of new technologies in the classroom has spurred a debate on the advantages and pitfalls that tools like Dynamic Geometry Software (DGS) or Computer Algebra Systems (CAS) are bringing in the teaching of Mathematics. The authors of this paper have been involved in exploring new automatic reasoning tools (see for example [10], [16]), mainly implemented in an experimental version of the software GeoGebra (GG), GeoGebra Discovery (GGD), maintained by Kovács Zoltán ([8]). They also want to convey the idea that these new technologies can help students to gain understanding and stimulate their curiosity when confronting geometric challenges.

With this intention in mind, the first author stepped on a Geometry problem posed in the 60th Spanish Mathematical Olympiad (OME):

Problem 1 *Let ABC be an scalene triangle and P a point in its interior such that $\angle PBA \cong \angle PCA$. Lines PB and PC intersect the interior and exterior angle bisectors through A at points Q and R respectively. Let S be the point such that $CS \parallel AQ$ and $BS \parallel AR$. Show that Q, R, S are collinear.*

The solution to this problem can be found in the web site of the LX OME ([13], [14]), but from the statement of this problem a related question arises concerning a geometric locus:

Problem 2 *Let ABC be a triangle. Find the geometric locus L' of all points P such that $\angle PBA \cong \angle PCA$.*

Computing geometric loci using algebraic symbolic tools has been an active area of interest for the research team of the first author ([2, 3, 4, 5, 6]). The software GGD, through its command `LocusEquation( , )`, allows the automatic computation of loci in a wide variety of geometric constructions, and Problem 2 was an inviting proposal for checking again the power of the automatic reasoning tools implemented in GGD.

Let us remark, very roughly speaking (see more details in the mentioned references), that locus computation through Dynamic Geometry Systems can be approached in quite different ways: just plotting some positions of the tracing point that builds the locus; or describing the locus through a numerical equation obtained from a large collection of such positions and with some probabilistic assumptions; or finding a more precise -yet approximate- equation of the locus by performing some symbolic elimination algorithms depending on the numerical coordinates of the points involved in the figure; or considering symbolic coordinates for the points in the figure and then performing elimination, yielding in this way an exact locus equation.

In this context, as purely symbolic computations are sometimes not performing well, GeoGebra Discovery `LocusEquation( , )` algorithm chooses to output, by elimination, an equation with coefficients depending on the *numerical* coordinates of the points in the construction, so it is not possible to use in a straightforward way the automated, symbolic, reasoning tools of GeoGebra Discovery to check the purely symbolic validity of the output. In the next sections we will develop the specific approaches we have developed to deal with this, apparently, simple, but intriguing locus. Let us construct an arbitrary triangle ABC in GGD, and let us add an arbitrary point P. We can then consider the angles $\delta = \angle PBA$, $\epsilon = \angle PCA$ (in what follows, as is usual in elementary geometry, we will consider non-oriented angles, i.e. all angles will be non-negative and less than π). The command `LocusEquation( , )` can admit as parameters a Boolean expression f relating geometric elements of a construction, and a moving point P involved in f, producing as output the locus of points P making true the given Boolean expression (see [9] for more on this command and other automatic reasoning tools in GGD). In our case, solving Problem 2 within the GGD environment amounts to type the simple command

`LocusEquation(`$\delta == \epsilon$`, `P`)`

The output after introducing this command is shown in Figure 1a. We will call L this automatically obtained locus (to distinguish it from our target locus L'). Notice that dealing with angles through symbolic computation requires handling some subtle issues: one, the transcendental character of the notion of angle (with respect to the coordinates of defining points); two, the need to deal with signs—and thus with real algebraic geometry—if approaching angles through trigonometric functions such as sine, cosine, tangent, etc.

After trying different configurations for the triangle ABC and inspecting the graphic and algebraic expression (with numerical, approximate coefficients) of this locus we come up with several observations:

Observation 1 (O1): The locus L seems to be the union of a circumference L_1 and a hyperbola L_2.

Observation 2 (O2): The circumference L_1 seems to be the circumcircle of ABC.

Observation 3 (O3): The hyperbola L_2 seems to contain the vertices A, B, C.

As the reader can appreciate, the use of the software GGD has allowed us to quickly establish a sequence of conjectures almost from scratch. The appearance of the circumcircle is unsurprising, because of the elementary properties of inscribed angles in a circle. In any case, these initial observations stir up our interest in the problem and give rise to a variety of questions that can lead to further exploration:

Question 1 (Q1): Does L really solve Problem 2? That is, $L \cong L'$?

Question 2 (Q2): If our previous observations O1-O3 are correct, the circumference L_1 and the hyperbola L_2 meet in general at four points, being A, B, C three of them. What can we say about the fourth point of intersection?

Question 3 (Q3): How can we characterize/construct the elements of the hyperbola L_2 (center, radius, axes, asymptotes, vertices and foci)?

Question 4 (Q4): Can we say anything else about this locus?

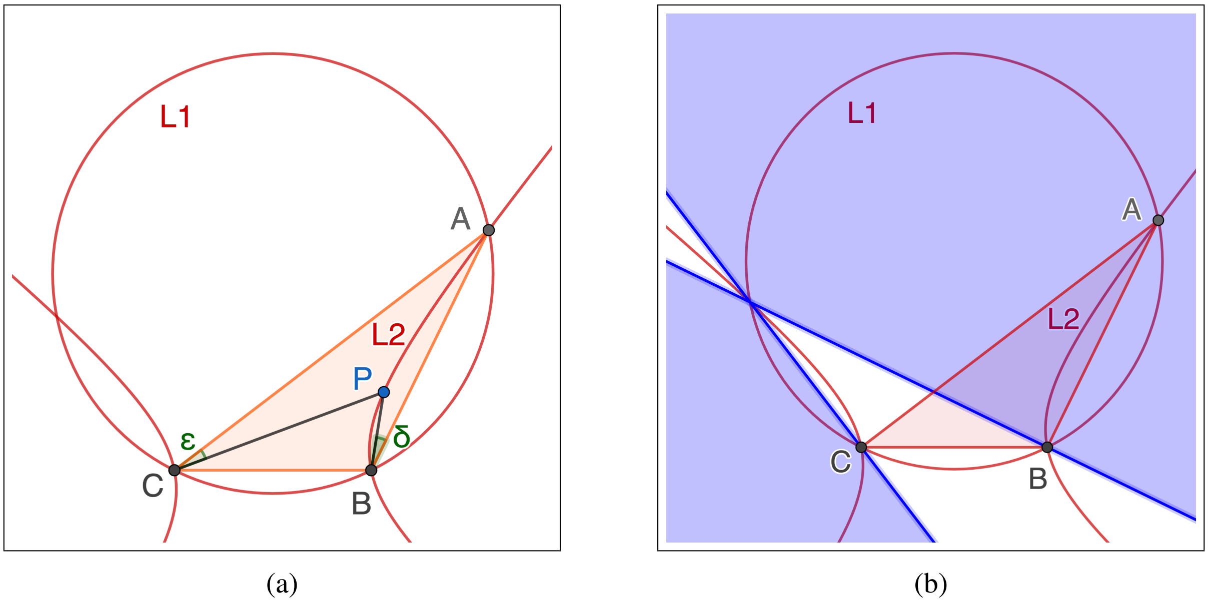


Figure 1: (a) Initial construction and locus L; (b) Fixing up locus L

2 Proving observations O1-O3

As a first consideration, the fact that vertices A, B and C belong to the locus arises some concern, because when point P coincides with any of these points the equality $\angle PBA = \angle PCA$ becomes degenerate. Moreover, some experimentation moving the point P along L reveals that on some arcs the equality $\angle PBA = \angle PCA$ seems to hold, while on others we rather have $\angle PBA + \angle PCA = \pi$. The reason behind this behavior comes from the fact that GGD handles the angle equality $\delta = \epsilon$ in symbolic computation terms, an involved approach, as already remarked. To see this, let us choose appropriate coordinates to simplify calculations. Since angles are preserved by homotheties and isometries, we can assume $B = (1, 0)$, $C = (-1, 0)$. Let us set $A = (a_1, a_2)$, $P = (x, y)$. Now, the equality $\delta = \epsilon$, by using the scalar product formula, becomes:

$$\delta = \epsilon \quad \Leftrightarrow \quad \cos\delta = \cos\epsilon \quad \Leftrightarrow \quad \frac{\vec{BA}\cdot\vec{BP}}{\|\vec{BA}\|\cdot\|\vec{BP}\|} = \frac{\vec{CA}\cdot\vec{CP}}{\|\vec{CA}\|\cdot\|\vec{CP}\|}$$

By squaring both sides of the last equality and removing denominators we finally get the polynomial identity

$$\begin{aligned} f :=& (\vec{BA}\cdot\vec{BP})^2(\|\vec{CA}\|\cdot\|\vec{CP}\|)^2 - (\vec{CA}\cdot\vec{CP})^2(\|\vec{BA}\|\cdot\|\vec{BP}\|)^2 \\ =& -(a_2y + (a_1+1)(x+1))^2(a_2^2 + (a_1-1)^2)(y^2 + (x-1)^2) \\ &+ (a_2y + (a_1-1)(x-1))^2(a_2^2 + (a_1+1)^2)(y^2 + (x+1)^2) = 0 \end{aligned}$$

Notice that the squaring process allows the possibility $\cos\delta = -\cos\epsilon$, and this means we can also have $\delta + \epsilon = \pi$ (this will bring us to Question Q1 later). The left side can be factorized (the GG CAS can perform this easily) to give us

$$f = -4(a_1^2y - x^2a_2 - y^2a_2 + ya_2^2 - y + a_2)(a_1^2xy - a_1x^2a_2 + a_1y^2a_2 + a_1a_2 - xya_2^2 - xy). \quad (1)$$

Let us consider the equations

$$l_1(x,y) := -a_2x^2 - a_2y^2 + (a_1^2 + a_2^2 - 1)y + a_2 = 0, \tag{2}$$
$$l_2(x,y) := -a_1a_2x^2 + a_1a_2y^2 + (a_1^2 - a_2^2 - 1)xy + a_1a_2 = 0. \tag{3}$$

Under the assumption $a_1a_2 \neq 0$ (which amounts to say that triangle ABC is non-degenerate and A is not contained in the Y-axis), these equations correspond to a circumference and a hyperbola, so that we have just verified O1, with L_1 and L_2 corresponding respectively to $l_1 = 0$ and $l_2 = 0$. When A lies on the Y-axis the hyperbola L_2 degenerates into the union of the coordinate axes, and from now on we will be assuming this is not the case, which is easy to handle. It is also straightforward to check now that $l_1 = 0$ on A, B and C, so that $l_1 = 0$ is the circumcircle of triangle ABC, as well as to check that the hyperbola $l_2 = 0$ contains the vertices of the triangle. Therefore, observations O1-O3 are correct.

3 Answering questions Q1-Q4

3.1 Answering Questions 1 and 2

In the previous section we already mentioned that, strictly speaking, the locus L' we are looking for does not coincide with L, because of two reasons:

i) The vertices B, C are degenerate points in the sense that equality $\delta = \epsilon$ loses its meaning;

ii) for certain points $P \in L$ we might have $\delta + \epsilon = \pi$ instead of $\delta = \epsilon$.

In order to clarify this last assertion we can proceed as follows: To preserve the equality $\cos\delta = \cos\epsilon$ we can add to the above locus equation $f = 0$ an extra condition (carrying our computations into the realm of computational real algebraic geometry, of greater complexity) which avoids the problem of loosing control on the signs of these cosines after squaring, that is

$$(\vec{BA} \cdot \vec{BP})(\vec{CA} \cdot \vec{CP}) \geq 0,$$

which insures the sign equality for both cosines. In coordinates this translates into

$$(a_2y + (a_1+1)(x+1))(a_2y + (a_1-1)(x-1)) \geq 0. \tag{4}$$

The left side is a product of two linear expressions in x, y. The first one, when equated to zero, corresponds to a line that contains vertex $C = (-1, 0)$, while the second one corresponds to a line containing vertex $B = (1, 0)$. Besides, by solving the system

$$\begin{cases} a_2y + (a_1+1)(x+1) = 0 \\ a_2y + (a_1-1)(x-1) = 0 \end{cases}$$

we obtain that these lines intersect at the point

$$D = \left(-a_1, \frac{a_1^2 - 1}{a_2}\right)$$

But it is easy to verify that this point D also belongs to the circle L_1 and to the hyperbola L_2, and so we have encounter the answer for Question 2, which asked for the fourth point of intersection of $L - 1$ and L_2. Notice also that vertex $A = (a_1, a_2)$ satisfies inequality (4), and all this lead us to the following answer to Question 1 (see Figure 1b):

Proposition 3 *The solution locus for Problem 2 is formed by two arcs: The circular arc BAC containing vertex A, together with a hyperbolic arc BAC (with two branches, since it crosses the line at infinity) containing A. We must exclude the vertices B, C from this locus.*

What else can we say about the point D? After some experimentation in the graphic view of GGD, we establish a conjecture to work with: *The fourth intersection point of $L_1 \cap L_2$ is the symmetric of point A with respect to the circumcenter of $\triangle ABC$.* Indeed, it is a simple exercise in coordinate geometry to find the coordinates of a point satisfying this condition and we retrieve again the very same coordinates of D. Hence, our claim is correct.

Since we already have four points of the hyperbola L_2, and having in mind that five points determine a conic, it is natural to look for a fifth point in L_2. If we think of the triangle centers, the orthocenter E of $\triangle ABC$ comes up as a suitable candidate, since $\angle EBA, \angle ECA \in \left\{\frac{\pi}{2} - \alpha, \frac{\pi}{2} + \alpha\right\}$, where $\alpha = \angle BAC$, and so they are either equal or supplementary. To compute the coordinates of E we can proceed by finding the equations of the altitudes of $\triangle ABC$ from A and C and solving the system of equations they form. The altitude from A has equation $x = a_1$, and the altitude from $C = (-1, 0)$, since $\overrightarrow{BA} = (a_1 - 1, a_2)$, is given by $(a_1 - 1)(x + 1) + a_2 y = 0$. Therefore,

$$\begin{cases} x = a_1 \\ (a_1 - 1)(x + 1) + a_2 y = 0 \end{cases} \longrightarrow \quad E = \left(a_1, \frac{1 - a_1^2}{a_2}\right),$$

and we realiza that E is the symmetric point of D with respect to the midpoint O of BC (!). So, we come up with these two facts:

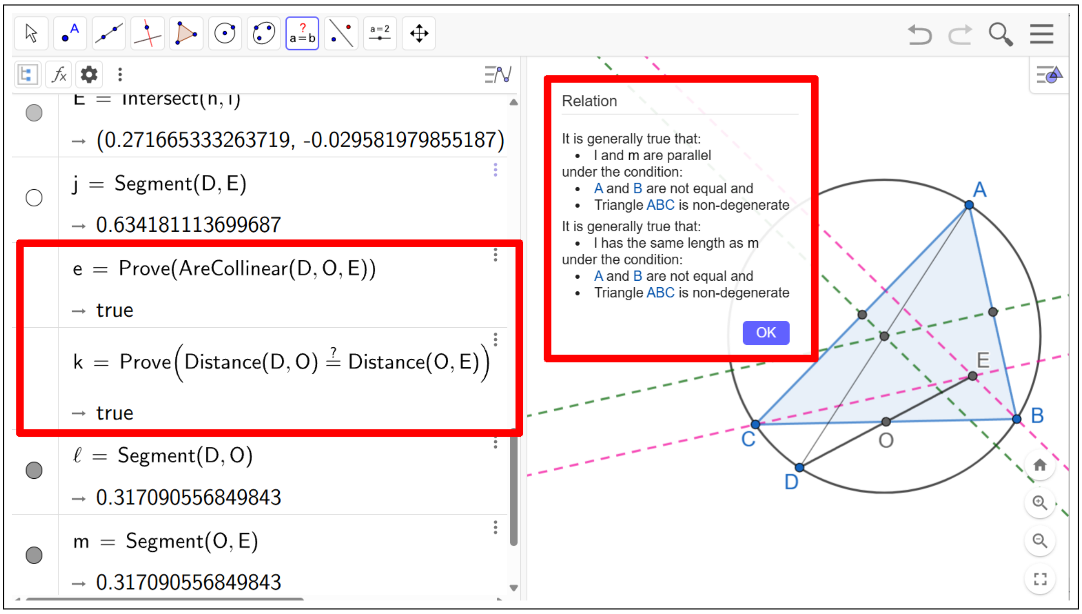


Figure 2: Use of GGD command `Prove()` and GGD Tool `Relation()`.

Proposition 4 *The hyperbola L_2 contains the vertices of the triangle, its orthocenter and the symmetric point D of A with respect to the circumcenter.*

Proposition 5 *In $\triangle ABC$, the symmetric point of a vertex with respect to the circumcenter and the orthocenter are symmetric with respect to the midpoint of its opposite side.*

Under a suitable construction in GGD, we can also verify the truth of this last statement by using the GGD commands `Prove(AreCollinear(D,O,E))` and `Prove(DO==OE)` (or the more informative command `ProveDetails()`), see Figure 2. We also invite the reader to try, in the latest version of GGD, the command `ShowProof()` as in `ShowProof(AreCollinear(D,O,E))`, which delivers in the CAS view a step-by-step proof of the geometric statement “The points D, O, E are collinear” (see Figure 3 and Appendix 1).

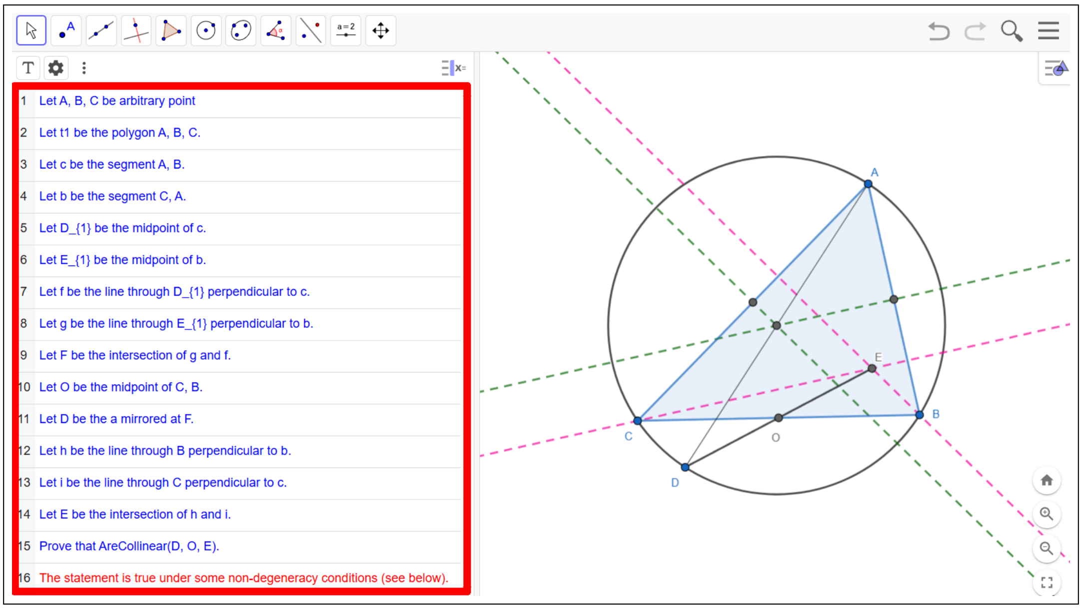


Figure 3: Use of GGD `ShowProof()` command.

3.2 Answering Question 3

The powerful tools included in the GeoGebra software allow us to explore easily the hyperbola L_2 and represent its main elements: center, foci, vertices, asymptotes and axes (see Figure 4a and [17, First Construction]). In particular, this facilitates the task of detecting invariant properties. So, after playing a while by dragging vertex A, we guess that

i) The center of the hyperbola is the midpoint of BC.

ii) The asymptotes are parallel to the angle bisectors at A.

The first assertion readily follows from the expression of the center of a conic in terms of the coefficients of its equation (see for example [1]). For the second assertion, we start by obtaining the equations of the angle bisectors at A. Let $Q = (x, y)$ be a point on one of the bisectors at A. Then

$$\cos(\angle BAQ) = \pm\cos(\angle CAQ) \quad \Leftrightarrow \quad \left(\frac{\vec{AB}\cdot\vec{AQ}}{\|\vec{AB}\|\cdot\|\vec{AQ}\|}\right)^2 = \left(\frac{\vec{AC}\cdot\vec{AQ}}{\|\vec{AC}\|\cdot\|\vec{AQ}\|}\right)^2$$

and we obtain the polynomial equation

$$\begin{aligned} g(x,y) :=& -4a_2(-a_1a_2x^2 + a_1a_2y^2 + (a_1^2 - a_2^2 - 1)xy \\ &+ (a_2^3 + a_1^2a_2 + a_2)x - (a_1^3 + a_1a_2^2y - a_1)y - a_1a_2) = 0. \end{aligned} \tag{5}$$

Equating to zero the second degree part $g_1(x, y) := -a_1a_2x^2 + a_1a_2y^2 + (a_1^2 - a_2^2 - 1)xy$ of the left hand side gives us the parallel lines to the angle bisectors through the origin. But g_1 does coincide with the second degree part in (3), which gives the asymptotes r_1, r_2 of L_2. From all this we infer that L_2 is an equilateral hyperbola, and the axes of L_2 are the bisectors of the right angles formed by r_1 and r_2.

In regard to the vertices and foci of L_2, no simple expressions for its coordinates were found, and the GGD CAS could not provide direct explicit output for them in our trials. Since the center of L_2 is at the origin of coordinates and the points $(x, y) \in L_2$ such that the their normal line contains the origin are only the vertices of L_2, these are determined by the real solutions of the system

$$\begin{cases} l_2(x,y) = 0 \\ \dfrac{\partial l_2}{\partial y}\Big/\dfrac{\partial l_2}{\partial x} = y/x \end{cases} \quad \longrightarrow \quad \begin{cases} -a_1a_2x^2 + a_1a_2y^2 + (a_1^2 - a_2^2 - 1)xy + a_1a_2 = 0 \\ \dfrac{2a_1a_2y + (a_1^2 - a_2^2 - 1)x}{-2a_1a_2x + (a_1^2 - a_2^2 - 1)y} = \dfrac{y}{x} \end{cases}$$

Solving this system leads to cumbersome expressions, but we can simplify them by setting (we assume $a_1a_2 \neq 0$)

$$\frac{a_1^2 - a_2^2 - 1}{a_1a_2} =: k,$$

so that the system becomes

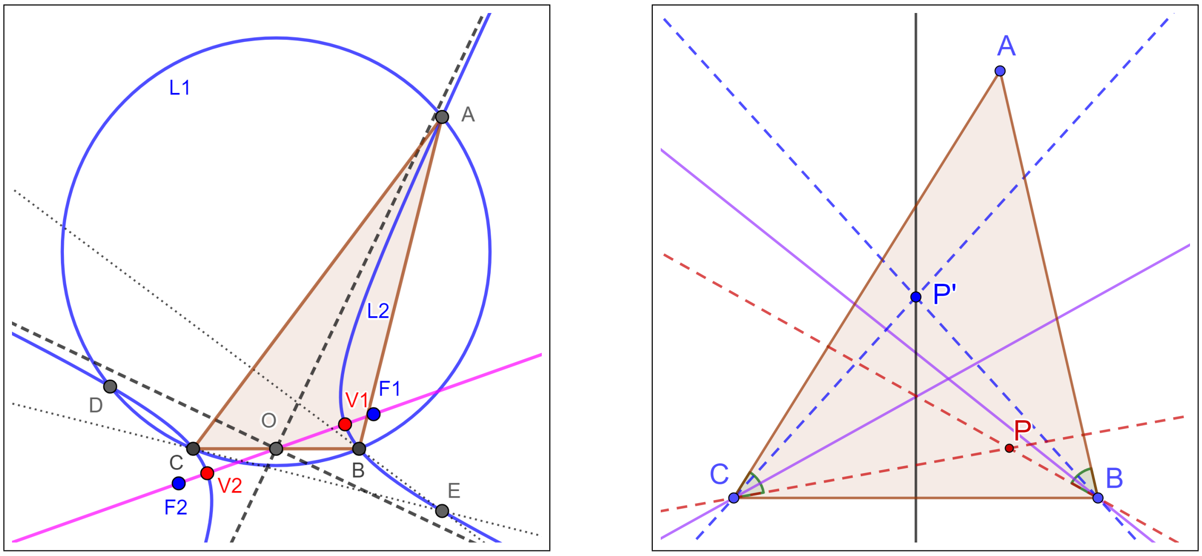
$$\begin{cases} -x^2 + y^2 + kxy + 1 = 0 \\ kx^2 - ky^2 + 4xy = 0 \end{cases}.$$

Even in this simpler form, the solutions of the system are more complex than expected and a CAS software can prove useful to help performing the computations: Setting

$$l_1 = \frac{\sqrt{(k^2+4)(-2+\sqrt{k^2+4})}}{k^2+4}, \quad l_2 = -\frac{\sqrt{(k^2+4)(-2+\sqrt{k^2+4})}}{k^2+4},$$

the real solutions that represent the coordinates of the vertices of L_2 can be expressed as follows:

$$\begin{cases} x_i = -\dfrac{((k^2+4)l_i^2 + 4)l_i}{k} \\ y_i = l_i \end{cases}, \quad i = 1, 2.$$



(a) (b)

Figure 4: (a) The hyperbola L_2 and its elements; (b) Isogonal conjugation

3.3 Answering Question 4

By using GGD we have been able to determine many properties of the locus L arising from Problem 2, but perhaps we can try to delve deeper and gain a better understanding of its nature. And here the human intuition still plays a role, sometimes based in prior knowledge, sometimes based in... luck? Consider the lines BP and CP, where P is such that $\angle PBA \cong \angle PCA = \delta$, and construct the symmetric lines BP' and CP' with respect to the internal angle bisectors at $\angle ABC$ and $\angle BCA$ respectively, which intersect at P'. It is straightforward that $\angle P'BC \cong \angle P'CB = \delta$, so that $\triangle P'BC$ is isosceles with $P'B \cong P'C$, and therefore P' lies in the perpendicular bisector of BC. In other words, the transformation $P \mapsto P'$ takes points in the hyperbola to points in the perpendicular bisector of BC, and viceversa. But this transformation has a name in plane geometry: *isogonal conjugation (with respect to a triangle)*, see for example [7]. In this context, a new statement concerning the hyperbola L_2 arises which makes clearer its appearance in our initial approach, and allows for an easy way to geometrically construct points of L_2 from points in the perpendicular bisector of BC (see [17, Second Construction], where by dragging point P' you can get points in L_2):

Proposition 6 *The hyperbola L_2 is the isogonal conjugate of the perpendicular bisector of the side BC.*

It is worthwhile to mention here that for well trained Geometry students (of the kind that receive intensive training for participating in mathematical contests such as the International Mathematical Olympiad) this connection to isogonal conjugacy could have been perceived in an initial exploration of Problem 2, and from this initial realization many of the properties deduced above become consequences of the isogonal conjugation properties. But, in general, isogonal transformations do not form part of the standard mathematics curriculum in secondary education, and our initial problem can become a starting point that leads to beautiful Geometry new topics for the interested students. The isogonal approach allows this informal interpretation of L: Let us consider the locus $\hat{L}$ of points whose distance to vertices B, C are equal. Usually, we infer that $\hat{L}$ is the perpendicular bisector l of BC. But if we think in projective terms and allow points at infinity, we could say that $\hat{L}$ also contains the line at infinity l_∞, since a point in infinity has the same distance from B and C (which is ∞!). And the isogonal conjugate of $\hat{L} = l_\infty \cup l$ is precisely $L = L_1 \cup L_2$ because from well-known properties of this transformation the isogonal conjugate of the line at infinity is the circumcircle of $\triangle ABC$!

4 Tricks and treats: Two constructions

As a final treat, we propose to the readers two constructions which came up along this work and allow for further exploration and reflection. In Appendix 2 we show why these constructions actually work.

FIRST CONSTRUCTION: One of the simplest constructions we devised to construct points of L_2 is represented in Figure 5a (see also [17, Third Construction]). Place a point D on the perpendicular bisector m of BC. Construct the circumference BCD and the internal angle bisector at A. Draw a parallel to this angle bisector through D, which intersects circumference BCD at other point P. Show that the locus of points P as D moves along m is the hyperbola L_2.

SECOND CONSTRUCTION: Notice that we used the powerful commands of GG to represent notable points such as the foci and vertices of L_2, but we did not actually gave a geometric construction of them. Here we propose a construction of the vertices of L_2 (since this hyperbola is equilateral, getting the foci from the vertices is straightforward) and we invite the readers to check its validity. Construct parallels through O to the angle bisectors at A. These parallels divide the plane in four right angles. Trace the bisectors of these angles b_1, b_2 and find the symmetric points B' and B'' of B with respect to these lines. Trace the circles c_1, c_2 with diameters $B'C$ and $B''C$. Now we claim (see Figure 5b and [17, Fourth Construction]):

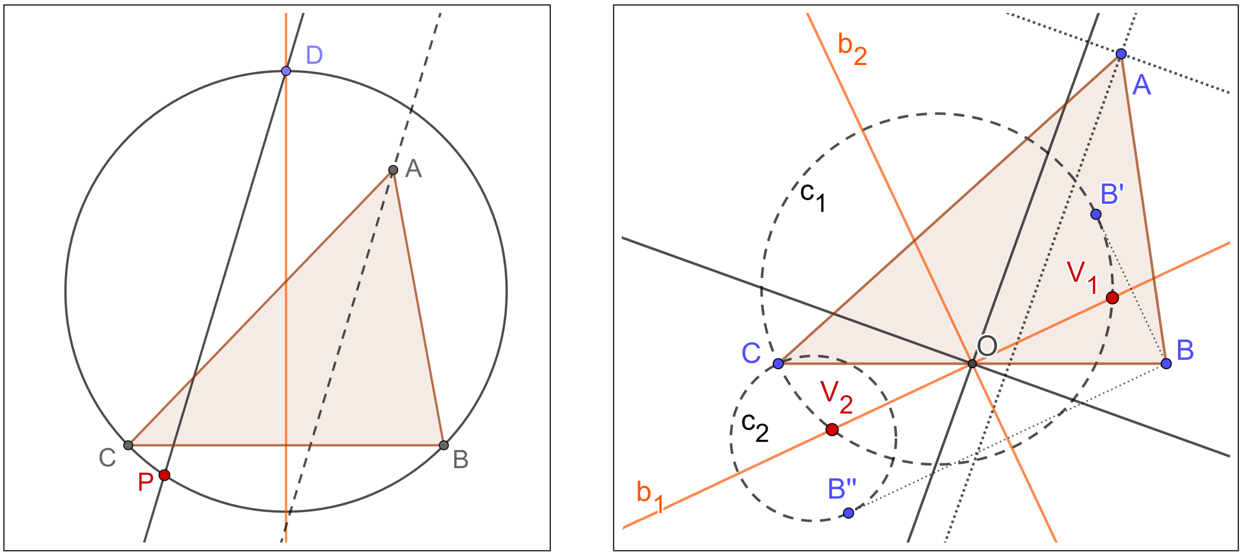
- One of the bisectors b_1, b_2 intersects both circles c_1, c_2.
- One of the circles c_1, c_2 meets both bisectors b_1, b_2.
- The intersection points of the circle meeting both bisectors with the bisector meeting both circles determine the vertices of L_2.

(Let us remark, again, that in the above we assume A does not lie in the Y-axis).

5 Conclusion

In summary, we think that we have shown how an apparently trivial geometric question: to describe the locus of points P such that $\angle PBA \cong \angle PCA$, for a triangle ABC, can give rise to a collection of—each time more involved—observations, then conjectures, and finally, mathematical statements.

What is notorious here is to remark that the use of a technological tool such as GeoGebra Discovery, that provides simultaneously plotting devices, dynamic geometry dragging possibilities, automatic answers for loci equations, computer algebra manipulation of the obtained equations. . . , fosters our imagination, and demands insistently to launch a joint cooperation with our mind to successfully deal with such problems.



(a) (b)

Figure 5: (a) Construction of points of L_2; (b) Construction of vertices of L_2

This is, we think, the main message of our contribution, the great possibilities provided by the technological tools, if we organize to use them "to do better things, instead of for just doing (old) things better" [1].

6 Acknowledgements

First author is partially supported by the project "Inteligencia aumentada en educación matemática mediante modelización, razonamiento automático e inteligencia artificial (IAxEM-CM" (PHS-2024/PH-HUM-383, granted by the *Comunidad de Madrid-Consejería de Educación, Ciencia y Universidades–Programas de Actividades de I+D entre Grupos de Investigación de la Comunidad de Madrid, Convocatoria Procesos Humanos y Sociales*.

Appendix 1

Here we show the output obtained with the command `ShowProof(AreCollinear(D,O,E))` of GGD mentioned in Subsection 3.1.

Let A, B, C be arbitrary point
Let t1 be the polygon A, B, C.
Let c be the segment A, B.
Let b be the segment C, A.
Let D_1 be the midpoint of c.
Let E_1 be the midpoint of b.
Let f be the line through D_1 perpendicular to c.
Let g be the line through E_1 perpendicular to b.
Let F be the intersection of g and f.
Let O be the midpoint of C, B.

[1]Freely quoting a sentence stated by Prof. J. Kaput, time ago, back in 1996, at ICME 8, see https://web.archive.org/web/20060621140909/http://mathforum.org/mathed/seville/followup.html

Let D be the a mirrored at F.
Let h be the line through B perpendicular to b.
Let i be the line through C perpendicular to c.
Let E be the intersection of h and i.
Prove that AreCollinear(D, O, E).
The statement is true under some non-degeneracy conditions (see below).
We prove this by contradiction.
Let free point A be denoted by (v1,v2).
Let free point B be denoted by (v3,v4).
Let free point C be denoted by (v5,v6).
Considering definition D_1 = Midpoint(c):
Let dependent point D_1 be denoted by (v7,v8).
$e1 := -v1 - v3 + 2 * v7 = 0$
$e2 := -v2 - v4 + 2 * v8 = 0$
Considering definition E_1 = Midpoint(b):
Let dependent point E_1 be denoted by (v9,v10).
$e3 := -v1 - v5 + 2 * v9 = 0$
$e4 := -v2 - v6 + 2 * v10 = 0$
Object f introduces the following extra variables:
v11: x value of an implicitly introduced second point for orthogonal line at D_1 to c
v12: y value of an implicitly introduced second point for orthogonal line at D_1 to c
$e5 := -v1 + v3 + v8 - v12 = 0$
$e6 := v2 - v4 + v7 - v11 = 0$
Object g introduces the following extra variables:
v13: x value of an implicitly introduced second point for orthogonal line at E_1 to b
v14: y value of an implicitly introduced second point for orthogonal line at E_1 to b
$e7 := v1 - v5 + v10 - v14 = 0$
$e8 := -v2 + v6 + v9 - v13 = 0$
Considering definition F = Intersect(g, f):
Let dependent point F be denoted by (v15,v16).
$e9 := v10 * v13 - v9 * v14 - v10 * v15 + v14 * v15 + v9 * v16 - v13 * v16 = 0$
$e10 := v8 * v11 - v7 * v12 - v8 * v15 + v12 * v15 + v7 * v16 - v11 * v16 = 0$
Considering definition O = Midpoint(C, B):
Let dependent point O be denoted by (v17,v18).
$e11 := -v3 - v5 + 2 * v17 = 0$
$e12 := -v4 - v6 + 2 * v18 = 0$
Considering definition D = Mirror(A, F):
Let dependent point D be denoted by (v19,v20).
$e13 := -v1 + 2 * v15 - v19 = 0$
$e14 := -v2 + 2 * v16 - v20 = 0$
Object h introduces the following extra variables:
v21: x value of an implicitly introduced second point for orthogonal line at B to b
v22: y value of an implicitly introduced second point for orthogonal line at B to b
$e15 := v1 + v4 - v5 - v22 = 0$
$e16 := -v2 + v3 + v6 - v21 = 0$
Object i introduces the following extra variables:
v23: x value of an implicitly introduced second point for orthogonal line at C to c
v24: y value of an implicitly introduced second point for orthogonal line at C to c
$e17 := -v1 + v3 + v6 - v24 = 0$
$e18 := v2 - v4 + v5 - v23 = 0$
Considering definition E = Intersect(h, i):
Let dependent point E be denoted by (v25,v26).
$e19 := v4 * v21 - v3 * v22 - v4 * v25 + v22 * v25 + v3 * v26 - v21 * v26 = 0$
$e20 := v6 * v23 - v5 * v24 - v6 * v25 + v24 * v25 + v5 * v26 - v23 * v26 = 0$
Thesis: AreCollinear(D, O, E), in algebraic form:
$T1 := v18 * v19 - v17 * v20 - v18 * v25 + v20 * v25 + v17 * v26 - v19 * v26 = 0$
Thesis reductio ad absurdum (denied statement):
v27: dummy variable to express negation
$(T1 * v27 - 1) = 0 \quad \rightarrow \quad (v27 * (((-v17) * v20) + (v17 * v26) + (v18 * v19) - (v18 * v25) - (v19 * v26) + (v20 * v25))) - 1 = 0$
$e21 := -1 + v18 * v19 * v27 - v17 * v20 * v27 - v18 * v25 * v27 + v20 * v25 * v27 + v17 * v26 * v27 - v19 * v26 * v27 = 0$
Without loss of generality, some coordinates can be fixed:
$\{v4 = 1, v3 = 0, v2 = 0, v1 = 0\}$
The statement can be suspected to be true under some non-degeneracy conditions:
Triangle ABC is non-degenerate
endg : $-1 - v5 * v28 = 0$
After substitutions:
sndg : $-1 - v5 * v28 = 0$
The statement requires some conditions:

A and B are not equal

All hypotheses and the negated thesis after substitutions:

$s1 : 2 * v7 = 0$

$s2 : -1 + 2 * v8 = 0$

$s3 : -v5 + 2 * v9 = 0$

$s4 : -v6 + 2 * v10 = 0$

$s5 : v8 - v12 = 0$

$s6 : -1 + v7 - v11 = 0$

$s7 : -v5 + v10 - v14 = 0$

$s8 : v6 + v9 - v13 = 0$

$s9 : v10 * v13 - v9 * v14 - v10 * v15 + v14 * v15 + v9 * v16 - v13 * v16 = 0$

$s10 : v8 * v11 - v7 * v12 - v8 * v15 + v12 * v15 + v7 * v16 - v11 * v16 = 0$

$s11 : -v5 + 2 * v17 = 0$

$s12 : -1 - v6 + 2 * v18 = 0$

$s13 : 2 * v15 - v19 = 0$

$s14 : 2 * v16 - v20 = 0$

$s15 : 1 - v5 - v22 = 0$

$s16 : v6 - v21 = 0$

$s17 : v6 - v24 = 0$

$s18 : -1 + v5 - v23 = 0$

$s19 : v21 - v25 + v22 * v25 - v21 * v26 = 0$

$s20 : v6 * v23 - v5 * v24 - v6 * v25 + v24 * v25 + v5 * v26 - v23 * v26 = 0$

$s21 : -1 + v18 * v19 * v27 - v17 * v20 * v27 - v18 * v25 * v27 + v20 * v25 * v27 + v17 * v26 * v27 - v19 * v26 * v27 = 0$

Now we consider the following equation:

$s1 * (-1/2 * v27 * v28 * v12 * v6^2 + 1/2 * v27 * v28 * v12 * v6 + 1/4 * v27 * v28 * v6^2 + v27 * v12 * v25 - 1/4 * v27 * v28 * v6 - 1/2 * v27 * v12 * v5 - 1/2 * v27 * v25 + 1/4 * v27 * v5) + s2 * (1/2 * v27 * v28 * v11 * v6^2 - 1/2 * v27 * v28 * v11 * v6 - v27 * v11 * v25 + 1/2 * v27 * v11 * v5) + s3 * (-1/2 * v27 * v28 * v14 * v6 + 1/4 * v27 * v28 * v6^2 + 1/2 * v27 * v28 * v14 - 1/4 * v27 * v28 * v6) + s4 * (1/2 * v27 * v28 * v13 * v6 - 1/4 * v27 * v28 * v6 * v5 - 1/2 * v27 * v28 * v13 + 1/4 * v27 * v28 * v5) + s5 * (-v27 * v28 * v15 * v6^2 + v27 * v28 * v15 * v6 + 2 * v27 * v15 * v25 - v27 * v15 * v5) + s6 * (v27 * v28 * v16 * v6^2 - v27 * v28 * v16 * v6 - 1/2 * v27 * v28 * v6^2 - 2 * v27 * v16 * v25 + 1/2 * v27 * v28 * v6 + v27 * v16 * v5 + v27 * v25 - 1/2 * v27 * v5) + s7 * (-v27 * v28 * v15 * v6 + 1/2 * v27 * v28 * v6 * v5 + v27 * v28 * v15 - 1/2 * v27 * v28 * v5) + s8 * (v27 * v28 * v16 * v6 - 1/2 * v27 * v28 * v6^2 - v27 * v28 * v16 + 1/2 * v27 * v28 * v6) + s9 * (-v27 * v28 * v6 + v27 * v28) + s10 * (-v27 * v28 * v6^2 + v27 * v28 * v6 + 2 * v27 * v25 - v27 * v5) + s11 * (-v27 * v16 + 1/2 * v27 * v26) + s12 * (v27 * v15 - 1/2 * v27 * v25) + s13 * (-v27 * v18 + v27 * v26) + s14 * (v27 * v17 - v27 * v25) + s15 * (-1/2 * v27 * v28 * v25 * v6 + 1/2 * v27 * v28 * v25) + s16 * (1/2 * v27 * v28 * v26 * v6 - 1/2 * v27 * v28 * v26 - 1/2 * v27 * v28 * v6 + 1/2 * v27 * v28) + s17 * (-1/2 * v27 * v28 * v25 * v6^2 + 1/2 * v27 * v28 * v6^2 * v5 + 1/2 * v27 * v28 * v25 * v6 - 1/2 * v27 * v28 * v6 * v5 - 2 * v27 * v15 * v25 + 2 * v27 * v15 * v5 + 1/2 * v27 * v25 * v5 - 1/2 * v27 * v5^2) + s18 * (1/2 * v27 * v28 * v26 * v6^2 - 1/2 * v27 * v28 * v6^3 - 1/2 * v27 * v28 * v26 * v6 + 1/2 * v27 * v28 * v6^2 + 2 * v27 * v15 * v26 - 2 * v27 * v15 * v6 - 1/2 * v27 * v26 * v5 + 1/2 * v27 * v6 * v5) + s19 * (-1/2 * v27 * v28 * v6 + 1/2 * v27 * v28) + s20 * (-1/2 * v27 * v28 * v6^2 + 1/2 * v27 * v28 * v6 - 2 * v27 * v15 + 1/2 * v27 * v5) + s21 * (-1) + sndg * (v27 * v15 * v6 + 1/2 * v27 * v25 * v6 - 1/2 * v27 * v6 * v5 - v27 * v15 - 1/2 * v27 * v25 + 1/2 * v27 * v5) \quad \rightarrow \quad 1 = 0$

Contradiction! This proves the original statement.

The statement has a difficulty of degree 5.

To get a better understanding of the inner workings of the command `ShowProof()` in GGD see [11].

Appendix 2

SKETCH OF PROOF FOR THE FIRST CONSTRUCTION

We have to show that for a point P defined as in the statement of the first construction, we must have $\angle ABP = \angle PCA$. Let us denote by α, β, γ the angles at A, B, C of $\triangle ABC$, and by δ, ϵ the angles

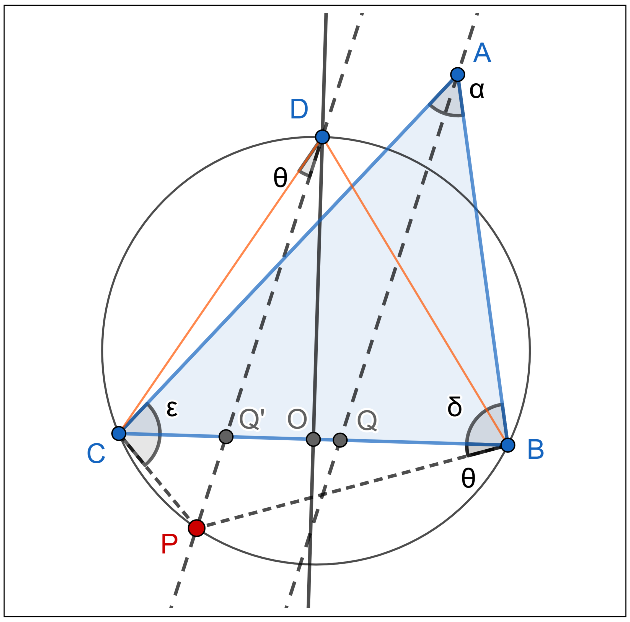


Figure 6: Geometric proof for Cosntruction 1.

$\angle ABP, \angle PCA$. We will assume that line DP intersects the segment CO. Set $\theta = \angle CDP$, and let us denote by Q the intersection of the interior bisector of α and BC and Q' the intersection of DP and BC. By angle chasing we arrive at the following identities (see Figure 6):

$$\delta = \angle ABP = \beta + \theta, \qquad \angle DQ'B = \angle AQB = \pi - \alpha/2 - \beta.$$

So, $\angle PCB = \angle PDB = 2\angle Q'DO + \theta = 2(\pi/2 - \angle DQ'B) + \theta = 2(\pi/2 - (\pi - \alpha/2 - \beta)) + \theta = -\pi + \alpha + 2\beta + \theta = \beta - \gamma + \theta$, and so $\epsilon = \angle PCA = \angle PCB + \gamma = \beta + \theta = \angle ABP = \delta$. We leave for the reader the study of other cases that depend on the location of the intersection of lines DP and BC.

An alternative approach can be made with coordinate geometry and some GG CAS assistance. Let us set $D = (0, d)$ and let us set coordinates $(0, b)$ for the center of the circumference through B, C, D. The equations for the parallels to the bisectors at A through D are given by

$$-a_1a_2x^2 + a_1a_2(y-d)^2 + (a_1^2 - a_2^2 - 1)x(y-d) = 0.$$

The circumference passing through B, C and D is given by the equation

$$x^2 + (y-b)^2 = b^2 + 1 \quad \longrightarrow \quad x^2 + y^2 - 2b - 1 = 0.$$

The relation between b and d is given by

$$(d-b)^2 = b^2 + 1 \quad \longrightarrow \quad d^2 - 2bd - 1 = 0.$$

By eliminating variables b, d in the system

$$\begin{cases} -a_1a_2x^2 + a_1a_2(y-d)^2 + (a_1^2 - a_2^2 - 1)x(y-d) = 0 \\ x^2 + y^2 - 2by - 1 = 0 \\ d^2 - 2bd - 1 = 0 \end{cases}$$

we are left with the equation that x, y, a_1, a_2 must satisfy. This equation can be obtained with the GG CAS command

$$\texttt{Eliminate}(\{-a_1a_2x^2 + a_1a_2(y-d)^2 + (a_1^2 - a_2^2 - 1)x(y-d) = 0,\\ x^2 + y^2 - 2by - 1 = 0, d^2 - 2bd - 1 = 0\}, \{b, d\})$$

which produces as output the equation

$$a_2a_1x^4 + a_2^2x^3y - a_1^2x^3y - a_2a_1x^2y^2 - a_2a_1x^2 + x^3y = 0.$$

Factorizing it we get

$$x^2(a_2a_1x^2 + a_2^2y - a_1^2xy - a_2a_1y^2 - a_2a_1 + xy) = -x^2l_2(x, y) = 0.$$

The points satisfying this equation are precisely those of the hyperbola L_2 plus those in the vertical axis $x = 0$ (which also appear, since the points D obviously satisfy the system).

Sketch of proof for the second construction

We have already seen that the parallels to the angle bisectors of $\angle BAC$ through the midpoint O of side BC are the asymptotes of the hyperbola L_2 (see Figure 5b). Since L_2 is an equilateral hyperbola, the bisectors b_1 and b_2 of the right angles formed by the asymptotes are the axes of the hyperbola. Therefore, L_2 is symmetric with respect to the lines b_1 and b_2, and the symmetric points B', B'' of B with respect to these lines do also belong to L_2. Also, C is symmetric with respect to O, and it is trivial to deduce that in fact $BB'CB''$ is a rectangle with sides parallel to b_1 and b_2, and inscribed in L_2. Depending on the position of the vertex A, the longest side of $BB'CB''$ can be either CB' or CB''. Assume CB' is the longest one. It is not hard to see that, in an equilateral hyperbola, this longest side must be parallel to its major axis, the one containing its vertices. The key fact which helps proving that Construction 2 actually works is:

Proposition 7 *Given an equilateral hyperbola, any circle which has as diameter a segment of the hyperbola parallel to its major axis passes through its vertices.*

The curious reader can check this statement by using coordinate geometry with the standard equation for an equilateral hyperbola $x^2 - y^2 = k^2$. In fact, this property is equivalent to stating that an equilateral hyperbola is the strophoid of a line (its minor axis) with respect to a fixed point (one of its vertices) and the perpendicular direction to the line (see [12, page 137]).

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